QUADRATICALLY LARGE REGULAR GRAPHS WITHOUT INTERNAL PARTITIONS

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Abstract. An internal partition of a graph is a nontrivial bipartition in which each vertex has at least as many neighbors in its own part as in the other part. In *Internal Partitions of Regular Graphs* (2016), Ban and Linial proposed in Conjecture 4 that every $2k$-regular graph on at least $4k$ vertices has an internal partition. We disprove Conjecture 4 at its proposed threshold. For every odd integer $k \geq 5$, we construct a $2k$-regular graph on exactly $\binom{k+2}{2}$ vertices with no internal partition. For every even integer $k \geq 8$, we construct a $2k$-regular graph on exactly $\binom{k+1}{2} + 2$ vertices with no internal partition. Both constructions also have connected complements. In fact, for each graph, every red–blue coloring in which each vertex has at least k neighbors of its own color is monochromatic.

1. Introduction

Let $G = (V, E)$ be a finite simple undirected graph. For $X \subseteq V$ and $v \in V$, write

$$d_X(v) := |N_G(v) \cap X|.$$

A bipartition $V = A \sqcup B$ is *internal* if A and B are nonempty and

$$d_A(v) \geq d_B(v) \quad \text{for every } v \in A, \qquad d_B(v) \geq d_A(v) \quad \text{for every } v \in B.$$

For a $2k$-regular graph, these inequalities simply require every vertex to have at least k neighbors in its own part.

Ban and Linial constructed $2k$-regular graphs on $3k + 2$ vertices with no internal partition and proposed Conjecture 4: every $2k$-regular graph on at least $4k$ vertices has an internal partition [2]. Equivalently, the conjecture asserts that every even-degree d-regular graph without an internal partition has fewer than $2d$ vertices. The constructions below have order $\Theta(d^2)$, exceeding the conjectured linear bound by a factor of $\Theta(d)$. Their complements are also connected, answering another question raised by Ban and Linial.

Theorem 1.1. *For every integer k that is either odd with $k \geq 5$ or even with $k \geq 8$, there exists a $2k$-regular graph G_k with no internal partition such that $\overline{G_k}$ is connected and*

$$|V(G_k)| = \begin{cases} \dbinom{k+2}{2}, & \text{if } k \text{ is odd,} \\ \dbinom{k+1}{2} + 2, & \text{if } k \text{ is even.} \end{cases}$$

Related work. Gerber and Kobler studied the same condition under the name *satisfactory graph partitioning* [6], and Ban and Linial surveyed related terminology and the regular-graph setting [2]. Linial and Louis proved that asymptotically almost every regular graph of any fixed even degree has an internal partition [7]. For degree 5, Bärnkopf et al. studied internal partitions and abelian Cayley graphs [3], while Csóka et al. later proved that asymptotically almost every 5-regular graph has an internal partition [4]. Anastos et al. obtained an approximate fixed-degree result allowing

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an arbitrarily small exceptional fraction [1]. Our growing-degree family refutes Ban and Linial's universal order threshold but does not address their fixed-degree conjecture [2].

2. Equivalent formulations

It is convenient to encode a bipartition by a red–blue coloring. For a coloring $c\colon V(G) \to \{\mathrm{red}, \mathrm{blue}\}$, define

$$d_c(v) = |\{u \in N_G(v) : c(u) = c(v)\}|.$$

Definition 2.1. A coloring of a $2k$-regular graph is *majority-stable* if $d_c(v) \geq k$ for every vertex v.

Proposition 2.2. *A $2k$-regular graph has an internal partition if and only if it has a nonmonochromatic majority-stable red–blue coloring.*

Proof. Color the two parts of a partition red and blue. Since every vertex has degree $2k$, the inequality between its same-part and opposite-part degrees is equivalent to having at least k same-colored neighbors. Requiring both parts to be nonempty is equivalent to requiring the coloring to be nonmonochromatic. □

The next observation is the two-vertex case of the clique type-class lemma of Gaikwad, Maity, and Tripathi [5]. We record it because it isolates the local mechanism used in the construction.

Lemma 2.3 (Adjacent-twin lemma)**.** *Let u and v be adjacent vertices in a $2k$-regular graph, and suppose that*

$$N(u) \setminus \{v\} = N(v) \setminus \{u\}.$$

Then u and v receive the same color in every majority-stable red–blue coloring.

Proof. The common external neighborhood has $2k - 1$ vertices. If u were red and v blue, then u would need at least k red vertices in that neighborhood and v would need at least k blue vertices there. This would require at least $2k$ vertices in a set of order $2k - 1$, a contradiction. □

3. The odd construction

The odd construction combines adjacent-twin pairs with an independent set and a chain of k-vertex layers. The terminal layer distributes the remaining pair-neighbors through cyclic intervals.

Fix an odd integer $k \geq 5$, and put

$$M = \frac{k+1}{2}, \qquad L = M - 2.$$

Then $L \geq 1$. All coordinate subscripts in this section are read modulo k.

3.1. Definition.

Take an independent set W of order $k + 1$, together with k two-vertex sets

$$P_i = \{x_i, y_i\}, \qquad i \in \mathbb{Z}/k\mathbb{Z}.$$

Let

$$B_t = \{b_{t,i} : i \in \mathbb{Z}/k\mathbb{Z}\}, \qquad 0 \leq t < L,$$

be pairwise disjoint sets of order k. Write

$$B_0 = A, \qquad C = B_{L-1}, \qquad c_i = b_{L-1,i}.$$

The edges of G_k are defined as follows.

1. Add the edge $x_i y_i$ for every $i \in \mathbb{Z}/k\mathbb{Z}$, with no edges between distinct pairs.
2. Join every vertex of W to both endpoints of every P_i.
3. Make $A = B_0$ a copy of K_k, and leave B_t independent for $1 \leq t < L$.
4. Join consecutive layers completely except along the coordinate matching: for $0 \leq t < L - 1$,

$$b_{t,i} b_{t+1,j} \in E(G_k) \quad \Longleftrightarrow \quad i \neq j.$$

(5) For every $i \in \mathbb{Z}/k\mathbb{Z}$ and every $0 \le t < L-1$, join both endpoints of P_i to $b_{t,i}$.
(6) For every $i \in \mathbb{Z}/k\mathbb{Z}$, join both endpoints of P_i to the M consecutive vertices of C, namely

$$c_i, c_{i+1}, \dots, c_{i+M-1}.$$

3.2. Order and regularity.

Proposition 3.1. *The graph G_k has $\binom{k+2}{2}$ vertices and is $2k$-regular.*

Proof. The order is

$$\begin{aligned} |V(G_k)| &= |W| + \sum_{i\in\mathbb{Z}/k\mathbb{Z}} |P_i| + \sum_{t=0}^{L-1} |B_t| \\ &= (k+1) + 2k + kL \\ &= (k+1) + 2k + k\left(\frac{k+1}{2} - 2\right) \\ &= \binom{k+2}{2}. \end{aligned}$$

We now verify that G_k is $2k$-regular. For $w \in W$,

$$\begin{aligned} d(w) &= \sum_{i\in\mathbb{Z}/k\mathbb{Z}} d_{P_i}(w) \\ &= \sum_{i\in\mathbb{Z}/k\mathbb{Z}} 2 \\ &= 2k. \end{aligned}$$

For $z \in P_i$, partitioning its neighborhood among the vertex families gives

$$\begin{aligned} d(z) &= d_W(z) + d_{P_i}(z) + \sum_{0\le t<L-1} d_{B_t}(z) + d_C(z) \\ &= (k+1) + 1 + (L-1) + M \\ &= 2k. \end{aligned}$$

Suppose $L > 1$. For a vertex $b_{0,i}$ of A,

$$\begin{aligned} d(b_{0,i}) &= d_A(b_{0,i}) + d_{B_1}(b_{0,i}) + d_{P_i}(b_{0,i}) \\ &= (k-1) + (k-1) + 2 \\ &= 2k. \end{aligned}$$

For $1 \le t < L-1$,

$$\begin{aligned} d(b_{t,i}) &= d_{B_{t-1}}(b_{t,i}) + d_{B_{t+1}}(b_{t,i}) + d_{P_i}(b_{t,i}) \\ &= (k-1) + (k-1) + 2 \\ &= 2k. \end{aligned}$$

Finally, every vertex of C lies in the cyclic intervals associated with exactly M pairs, so

$$\begin{aligned} d(c_i) &= d_{B_{L-2}}(c_i) + \sum_{j\in\mathbb{Z}/k\mathbb{Z}} d_{P_j}(c_i) \\ &= (k-1) + 2M \\ &= 2k. \end{aligned}$$

When $L = 1$, we have $C = A$, and the corresponding calculation is

$$\begin{aligned} d(c_i) &= d_A(c_i) + \sum_{j \in \mathbb{Z}/k\mathbb{Z}} d_{P_j}(c_i) \\ &= (k-1) + 2M \\ &= 2k. \end{aligned}$$

□

3.3. **No internal partition.**

Proposition 3.2. *The graph G_k has no internal partition.*

Proof. Let c be a majority-stable red–blue coloring of G_k. By Lemma 2.3, the two vertices of every pair P_i have the same color. Since $k = 2M - 1$, one color occurs on at least M pairs. After exchanging the color names if necessary, assume that this color is red.

Every vertex of W is red. Indeed, a blue vertex of W would have at most

$$2(M-1) = k-1$$

blue neighbors. Next, no pair can be blue. An endpoint of a blue pair has its blue mate, no blue neighbors in W, and only $(L-1) + M = k-2$ other neighbors. Its same-color degree is therefore at most

$$1 + (L-1) + M = k-1,$$

again contradicting majority stability. Thus W and every pair are red.

Every vertex of C has $2M = k+1$ red neighbors among the pairs. A blue vertex of C would consequently have at most $k-1$ blue neighbors, so C is red.

Case 1: $L = 1$. Then $A = C$, so every layer is red.

Case 2: $L > 1$. The base layer $B_{L-1} = C$ is red by the preceding paragraph. For $1 \le t \le L-2$, assume that B_{t+1} is red. A blue vertex of B_t has no blue coordinate-pair neighbors and no blue neighbors in B_{t+1}. Since B_t is independent, all its blue neighbors must lie in B_{t-1}, where it has only $k-1$ neighbors. Thus B_t is red. Applying this step for

$$t = L-2, L-3, \dots, 1$$

shows that every B_t with $1 \le t < L$ is red.

A blue vertex of $A = B_0$ now has no blue coordinate-pair neighbors and no blue neighbors in B_1. Its only possible blue neighbors are the other $k-1$ vertices of A, a final contradiction. Thus A is red.

In either case, every majority-stable coloring of G_k is monochromatic. □

3.4. **Connected complement.**

Proposition 3.3. *The complement $\overline{G_k}$ is connected.*

Proof. The set W is a clique in $\overline{G_k}$, and every vertex in every layer B_t is adjacent in $\overline{G_k}$ to every vertex of W. Hence W and all layer vertices lie in one component of the complement.

An endpoint of P_i is adjacent in G_k to only $M < k$ vertices of the terminal layer C. It is therefore adjacent in $\overline{G_k}$ to some vertex of C. Thus every pair vertex lies in the same component, and $\overline{G_k}$ is connected. □

4. The even construction

The even construction uses $k-1$ adjacent-twin pairs P_i and a chain of $(k-1)$-vertex blocks B_t, C, and D.

Fix an even integer $k \geq 8$, and put

$$q = k-1, \qquad M = \frac{k}{2}, \qquad L = M-2.$$

Pair indices and the coordinate subscripts on $b_{t,i}$, c_i, and d_i are read modulo q; subscripts on the vertices of W are ordinary integer indices.

4.1. **Definition.** Let

$$W = \{w_0, w_1, \ldots, w_k\}, \qquad P_i = \{x_i, y_i\} \quad (i \in \mathbb{Z}/q\mathbb{Z}).$$

There is one further vertex a, together with $L-2$ ordinary blocks

$$B_t = \{b_{t,i} : i \in \mathbb{Z}/q\mathbb{Z}\}, \qquad 0 \leq t < L-2,$$

and two final blocks

$$C = \{c_i : i \in \mathbb{Z}/q\mathbb{Z}\}, \qquad D = \{d_i : i \in \mathbb{Z}/q\mathbb{Z}\}.$$

Distinguish the vertices

$$e = c_0, \qquad f = d_0, \qquad j = d_{q-1}.$$

Thus $b_{t,i}$ is the vertex of coordinate i in B_t, while c_i and d_i are the vertices of coordinate i in the penultimate block C and terminal block D, respectively.

The edges of G_k are defined as follows.

(1) Add the edge $x_i y_i$ for every $i \in \mathbb{Z}/q\mathbb{Z}$, with no edges between distinct pairs.
(2) Join every vertex of W to both endpoints of every P_i.
(3) If $L > 2$, make $B_0 \cup \{a\}$ a copy of K_k. If $L = 2$, make $C \cup \{a\}$ a copy of K_k.
(4) If $L > 2$, join B_t and B_{t+1} completely for every $0 \leq t < L-3$, and join B_{L-3} completely to C.
(5) Join every vertex of $C \setminus \{e\}$ to every vertex of D, and add the edge ef.
(6) For every $i \in \mathbb{Z}/q\mathbb{Z}$ and every $0 \leq t < L-2$, join both endpoints of P_i to $b_{t,i}$. Also join both endpoints of P_i to c_i.
(7) For every $i \in \mathbb{Z}/q\mathbb{Z}$, join both endpoints of P_i to the M consecutive vertices of D, namely

$$d_i, d_{i+1}, \ldots, d_{i+M-1}.$$

(8) For every $0 \leq i < M$, join both endpoints of P_i to a. For every $M \leq i < q$, join both endpoints of P_i to e.
(9) Make W induce the path $w_0 w_1 \cdots w_k$, and make D induce the path $d_0 d_1 \cdots d_{q-1}$. Add the edges $w_0 a$ and $w_k j$.

4.2. **Order and regularity.**

Proposition 4.1. *The graph G_k has $\binom{k+1}{2} + 2$ vertices and is $2k$-regular.*

Proof. The order is

$$\begin{aligned}
|V(G_k)| &= |W| + \sum_{i\in\mathbb{Z}/q\mathbb{Z}} |P_i| + |\{a\}| + \sum_{0\le t<L-2} |B_t| + |C| + |D| \\
&= (k+1) + 2q + 1 + (L-2)q + q + q \\
&= 3k + L(k-1) \\
&= 3k + \left(\frac{k}{2} - 2\right)(k-1) \\
&= \binom{k+1}{2} + 2.
\end{aligned}$$

We now verify that G_k is $2k$-regular. Every $w \in W$ has two neighbors in $W \cup \{a, j\}$ and both endpoints of every pair as neighbors. Thus

$$\begin{aligned}
d(w) &= \sum_{i\in\mathbb{Z}/q\mathbb{Z}} d_{P_i}(w) + d_{W\cup\{a,j\}}(w) \\
&= 2q + 2 \\
&= 2k.
\end{aligned}$$

For $z \in P_i$, partitioning its neighborhood among the vertex families gives

$$\begin{aligned}
d(z) &= d_W(z) + d_{P_i}(z) + \sum_{0\le t<L-2} d_{B_t}(z) + d_C(z) \\
&\quad + d_{\{a,e\}}(z) + d_D(z) \\
&= (k+1) + 1 + (L-2) + 1 + 1 + M \\
&= 2k.
\end{aligned}$$

The copy of K_k containing a contributes $k-1$ neighbors to a. Together with its pair-neighbors and w_0, this gives

$$\begin{aligned}
d(a) &= (k-1) + \sum_{0\le i<M} d_{P_i}(a) + d_W(a) \\
&= (k-1) + 2M + 1 \\
&= 2k.
\end{aligned}$$

When $L > 2$, a vertex $b_{t,i}$ has q neighbors in the block following B_t. It has another q neighbors in B_{t-1} when $t > 0$, or among the other vertices of $B_0 \cup \{a\}$ when $t = 0$. Together with the two endpoints of P_i, this gives

$$\begin{aligned}
d(b_{t,i}) &= q + q + d_{P_i}(b_{t,i}) \\
&= q + q + 2 \\
&= 2k.
\end{aligned}$$

For $c_i \ne e$, there are q neighbors before D: these lie in B_{L-3} when $L > 2$ and among the other vertices of $C \cup \{a\}$ when $L = 2$. Therefore

$$\begin{aligned}
d(c_i) &= q + d_D(c_i) + d_{P_i}(c_i) \\
&= q + q + 2 \\
&= 2k.
\end{aligned}$$

The same count gives q such neighbors for e. Its remaining neighbors are f, both endpoints of P_0, and the endpoints of $P_M, \ldots, P_{q-1}$. Thus

$$\begin{aligned} d(e) &= q + d_{\{f\}}(e) + d_{P_0}(e) + \sum_{M \le i < q} d_{P_i}(e) \\ &= q + 1 + 2 + 2(M-1) \\ &= 2k. \end{aligned}$$

Finally, every vertex of D has two neighbors in $D \cup \{e, w_k\}$: the internal vertices have two path-neighbors, f has d_1 and e, and j has d_{q-2} and w_k. Since each vertex of D also has all $q-1$ vertices of $C \setminus \{e\}$ and the endpoints of exactly M pairs as neighbors,

$$\begin{aligned} d(d_i) &= d_{C\setminus\{e\}}(d_i) + \sum_{r \in \mathbb{Z}/q\mathbb{Z}} d_{P_r}(d_i) \\ &\quad + d_{D\cup\{e,w_k\}}(d_i) \\ &= (q-1) + 2M + 2 \\ &= 2k. \end{aligned}$$

□

4.3. **No internal partition.**

Proposition 4.2. *The graph G_k has no internal partition.*

Proof. Let c be a majority-stable red–blue coloring of G_k. By Lemma 2.3, every pair P_i is monochromatic. Since $q = 2M - 1$, one color occurs on at least M pairs. After exchanging the color names if necessary, assume that this color is red. At most $M - 1$ pairs are then blue.

Every vertex of W is red. Indeed, every vertex of W has exactly two neighbors outside the pairs. A blue vertex of W would have at most

$$2(M-1) = k-2$$

blue pair-neighbors, so both of its remaining neighbors would have to be blue. This condition propagates along the path $w_0 w_1 \cdots w_k$, making all of W blue. An endpoint of a red pair would then have no red neighbor in W and only $k-1$ neighbors outside W, contradicting majority stability.

Next, no pair can be blue. An endpoint of a blue pair has no blue neighbor in W and only $k-1$ neighbors outside W, again contradicting majority stability. Thus every pair is red.

Each vertex of D has $2M = k$ red pair-neighbors. If j were blue, all its remaining k neighbors would have to be blue, including w_k. This contradicts the fact that W is red, so j is red.

Now suppose that $0 \le r < q-1$ and d_{r+1} is red. If d_r were blue, then its k red pair-neighbors would force all its remaining k neighbors to be blue. This is impossible because d_{r+1} is one of those neighbors. Starting with $d_{q-1} = j$ and working backward shows that every vertex of D is red.

Let $c_i \ne e$. It has no blue neighbors in D and no blue coordinate-pair neighbors. If $L > 2$, its remaining neighbors lie in B_{L-3}; if $L = 2$, they lie among the other vertices of $C \cup \{a\}$. In either case there are only $q = k-1$ of them. Hence $C \setminus \{e\}$ is red.

The vertex e is adjacent to both red endpoints of P_0 and to the endpoints of the $M-1$ red pairs $P_M, \ldots, P_{q-1}$. It therefore has

$$2 + 2(M-1) = 2M = k$$

red pair-neighbors, together with the red neighbor f. Thus e is red, and all of C is red.

Case 1: $L = 2$. There are no ordinary blocks, and every neighbor of a is red. Hence a is red.

Case 2: $L > 2$. The base block C is red. For $0 \le t \le L-3$, assume that the block following B_t is red. A blue vertex of B_t has no blue coordinate-pair neighbors and no blue neighbors in the following block. Its only possible blue neighbors lie in B_{t-1} when $t > 0$, or among the other vertices of $B_0 \cup \{a\}$ when $t = 0$. In either case there are at most $q = k - 1$ such neighbors, so B_t is red. Applying this step for

$$t = L - 3, L - 4, \dots, 0$$

shows that every ordinary block is red. Every neighbor of a is therefore red, so a is red.

In either case, every majority-stable coloring of G_k is monochromatic. □

4.4. **Connected complement.**

Proposition 4.3. *The complement $\overline{G_k}$ is connected.*

Proof. The graph induced by W in $\overline{G_k}$ is the complement of a path on $k + 1 \geq 9$ vertices and is therefore connected. Every vertex outside $W \cup \bigcup_i P_i$ is adjacent in $\overline{G_k}$ to at least one vertex of W, so all nonpair vertices lie in the same component.

An endpoint of P_i is adjacent in G_k to exactly $M < q$ vertices of D. It is therefore adjacent in $\overline{G_k}$ to some vertex of D. Thus every pair vertex lies in the same component, and $\overline{G_k}$ is connected. □

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References

1. Michael Anastos, Oliver Cooley, Mihyun Kang, and Matthew Kwan, *Partitioning problems via random processes*, Journal of the London Mathematical Society **110** (2024), no. 6, e70010.
2. Amir Ban and Nati Linial, *Internal partitions of regular graphs*, Journal of Graph Theory **83** (2016), no. 1, 5–18.
3. Pál Bärnkopf, Zoltán Lóránt Nagy, and Zoltán Paulovics, *A note on internal partitions: The 5-regular case and beyond*, Graphs and Combinatorics **40** (2024), 36.
4. Endre Csóka, Panna Tímea Fekete, Zoltán Lóránt Nagy, and Levente Szemerédi, *Bisection width, max-cut and internal partitions of 5-regular graphs*, arXiv preprint arXiv:2509.08531, 2025.
5. Ajinkya Gaikwad, Soumen Maity, and Shuvam Kant Tripathi, *Parameterized complexity of satisfactory partition problem*, Theoretical Computer Science **907** (2022), 113–127.
6. Michael U. Gerber and Daniel Kobler, *Algorithmic approach to the satisfactory graph partitioning problem*, European Journal of Operational Research **125** (2000), no. 2, 283–291.
7. Nathan Linial and Sria Louis, *Asymptotically almost every 2r-regular graph has an internal partition*, Graphs and Combinatorics **36** (2020), no. 1, 41–50.

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